Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

The practical issues that arise due to the interaction between three principal players in any quantitative strategy, namely, the alpha model, the risk model and the constraints are collectively referred to as Factor Alignment Problems (FAP). While the role of misaligned alpha factors in causing FAP is relatively easy to understand, incorporating the impact of constraints entails considerable analytical complexity that most consultants and researchers find difficult to fathom. A few of them have even gone to the extent of suggesting that aligning alpha and risk factors should suffice in handling FAP. We provide a solid rebuttal to this line of thinking by demonstrating typical symptoms of FAP in optimal portfolios generated by using completely aligned alpha and risk models. Additionally, we provide theoretical guidance to clarify the role of constraints in influencing FAP and illustrate how the Alpha Alignment Factor (AAF) methodology can handle misalignment resulting from constraints, analytical complexities notwithstanding.
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

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1 Introduction

The practical issues that arise due to the interaction between three principal players in any quantitative strategy, namely, the alpha model, the risk model and the constraints are collectively referred to as Factor Alignment Problems (FAP). Examples of FAP include risk-underestimation of optimized portfolios, undesirable exposures to factors with hidden and unaccounted systematic risk, consistent failure in achieving ex-ante performance targets, and inability to harvest high quality alphas into above-average IR.

Despite several studies (Alford et al. (2003); Renshaw et al. (2006); Lee and Stefek (2008); Saxena and Stubbs (2010a,b); Ceria et al. (2012); Saxena and Stubbs (2011)), there is considerable disparity in understanding the sources of FAP. While the role of misaligned alpha factors is relatively easy to understand, incorporating the impact of constraints entails considerable analytical complexity that most consultants and researchers find difficult to fathom. A few of them have even gone to the extent of suggesting that aligning alpha and risk factors should suffice in handling FAP. We provide a solid rebuttal to this line of thinking by demonstrating typical symptoms of FAP in optimal portfolios generated by using completely aligned alpha and risk models. Additionally, we provide theoretical guidance to clarify the role of constraints in influencing FAP and illustrate how the Alpha Alignment Factor (AAF) methodology can handle misalignment resulting from constraints, analytical complexities notwithstanding.

The results presented in this paper are important for two reasons. One, by virtue of recent developments in risk model infrastructure such as the Risk Model Machine (RMM), it is now possible to surgically remove the misalignment between alpha and risk factors by constructing custom risk models (CRM) that explicitly incorporate the alpha factors. Construction of CRM involves complete recalibration of the covariance matrix by re-running the cross sectional regressions, recomputing factor returns attributed to the original and custom risk factors, and using the resulting time series of factor and residual returns to compute the factor-factor covariance matrix and specific risks. Among other things, CRM capture the temporal fluctuations in the volatility of alpha factors, and hence provide a sophisticated means of handling misalignment arising from alphas. Unfortunately, no such exact approach has yet been developed to handle misalignment arising from constraints which
is usually more difficult to characterize apriori. For instance, misalignment arising from the turnover constraint depends on the asset trades which are determined only during the portfolio construction phase. Our results show that despite their analytical complexity, the misalignment arising from constraints is a “real” problem that compromises the efficiency of the resulting portfolios. In other words, performance sensitive portfolio managers cannot afford to ignore risk under-estimation problems that arise due to constraints.

Second, we revisit the Alpha Alignment Factor (AAF) methodology and establish its efficacy in handling misalignment arising from constraints. While researchers have previously argued about the capability of the AAF methodology to handle misalignment arising from both alphas and constraints, a focused empirical illustration of AAF in handling FAP arising exclusively from constraints has been lacking until our work. To summarize, we not only highlight the gravity of misalignment problems that arise due to constraints but also illustrate a practical solution in the form of the AAF to circumvent them.

The rest of the paper is organized as follows. We initiate our investigation (Section 2) with a very simple strategy that has no conspicuous source of misalignment; specifically, we use an identical set of alpha and risk factors. Despite the idyllic nature of this strategy, the resulting optimal portfolios display the quintessential symptom of FAP, namely, downward bias in risk prediction. We investigate the source of the mentioned bias and trace it to the long-only and active asset bound constraints; while neither of these constraints have appeared prominently in the alignment debate, our results show that they can introduce statistically significant bias in risk prediction. Subsequently, we extend our findings to variants of the above strategy that encompasses practical considerations such as turnover limitations, average daily volume (ADV) constraint, illiquidity motivated asset bounds, market impact function, transaction cost models, etc.

Section 3 revisits the constrained Mean Variance Optimization (MVO) model from analytical perspective and dwells on the interchangeable role of alpha and constraints. Among other things, we give a structural result to demonstrate how the orthogonal components of both alpha and constraints can introduce misalignment, and misguide the optimizer to take exposure to latent systematic risk factors thus providing theoretical underpinning to results presented in this paper. In Section 4 we highlight the “opportunity” cost of not addressing the misalignment between constraints and risk factors. Our results show that the ill effects of misalignment due to constraints extend beyond the immediately visible effects such as biased risk prediction, and materially impact the efficiency of the resulting portfolios. In other words, the latent systematic risk factors associated with constraints disorient the ability of the optimizer to perform optimal budget and risk allocation as justified by the risk-return characteristics of individual securities. As a byproduct, we show how the Alpha Alignment Factor (AAF) approach remedies many of these problems and helps restore efficiency of the optimal portfolio. We end the paper with some concluding remarks in Section 5.

Throughout this paper we use the words expected returns and alpha synonymously. Also, the proofs of theoretical results presented in this paper are fairly straightforward and omitted for the sake of brevity.
2 Misalignment from Constraints

We initiate our investigation with a very simple strategy to highlight the role of constraints apropos FAP. We used the following long-only strategy, referred to as the base strategy, in our experiments.

\[
\begin{align*}
\text{maximize} & \quad \text{Expected Return} \\
\text{s.t} & \quad \text{Fully invested long-only portfolio} \\
& \quad \text{Active sector exposure constraint} \\
& \quad \text{Active industry exposure constraint} \\
& \quad \text{Active asset bounds constraint} \\
& \quad \text{Active Risk constraint } (\sigma) \\
& \quad \text{Benchmark } = \text{S&P 600.}
\end{align*}
\]

We used Axioma’s short horizon fundamental risk model (US2AxiomaSH) to define the tracking error constraint, and an equal weighted combination of “Growth” and “Short Term Momentum” factors in US2AxiomaSH to define the expected returns thus ensuring complete alignment between the alpha and risk factors. The industry factors in US2AxiomaSH were used to define the industry bound constraints whereas combinations of industry factors as dictated by GICS were used to define the sector bound constraints. We ran a monthly back-test using the above strategy in the 1998-2010 time period for \(\sigma = 0.5\%, 0.6\%, \ldots, 3.0\%\).

Next, we report our computational findings. Figure 1a plots the predicted and realized active risk of the portfolios for various risk target levels; for the sake of comparison, we also show a dotted line that corresponds to completely unbiased risk prediction. Figure 1b reports the same information using the concept of the bias statistic. The bias statistic is a statistical metric which is used to measure the accuracy of risk prediction; if the ex-ante risk prediction is unbiased, then the bias statistic should be close to 1.0 (see Saxena and Stubbs (2010a) for more details). Several comments are in order.

First, note the significant downward bias in risk prediction (Figure 1a); Figure 1b shows that the bias statistics associated with active risk forecasts are significantly above the 95% confidence interval thereby confirming the statistical significance of the mentioned bias. Second, the bias statistic associated with the total risk of the portfolio was within the 95% confidence interval [0.88, 1.12]. This suggests that the risk model was able to produce unbiased forecasts for the total risk of the portfolio, and the downward bias was limited to the active risk forecasts, a classic symptom of FAP (also see Saxena and Stubbs (2010a)). Third, by virtue of complete alignment between the alpha and risk models, it follows that the residual component of alpha, \(\alpha_{\perp}\), that is uncorrelated with factors in the risk model is vacuous\(^1\). Among other things, this implies that penalizing the exposure of the portfolio to \(\alpha_{\perp}\) will not remedy the problem. Fourth, even though \(\alpha_{\perp} = 0\), the residual component of

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\(1\)Given an arbitrary factor \(f\) and a set of risk factors \(S = \{f_1, f_2, \ldots, f_n\}\), the residual or orthogonal component of \(f\), denoted by \(f_{\perp}\), is defined to be the residual obtained by regressing \(f\) against factors in \(S\). The portion of \(f\), namely \(f - f_{\perp}\), which is explained by the risk factors in \(S\) is referred to as the spanned component of \(f\). Unless otherwise stated, we always assume that \(S\) contains all the risk factors in the risk model under consideration.
implied alpha, $\gamma_\perp$, can still be non-zero, have overlap with latent systematic risk factors and thus induce the optimizer to take inadvertent exposure to systematic risk factors.

One way of confirming the above hypothesis is to compute the realized systematic risk in $\gamma_\perp$ and compare it with the systematic risk in a median risk factor of US2AxiomaSH. We used the augmented regression methodology described in Saxena and Stubbs (2011) to compute the realized systematic risk in $\gamma_\perp$; Figure 1c reports the timeseries of annualized volatility of factor returns that can be attributed to $\gamma_\perp (\sigma = 2\%)$ computed using a rolling 24-period window. For the sake of comparison, we also report median systematic risk in (scaled) risk factors associated with US2AxiomaSH. Except for a brief period during the 2008 crisis, the two timeseries shown in Figure 1c track each other quite consistently. This implies that not only does $\gamma_\perp$ have systematic risk, $\gamma_\perp$ is, in fact, comparable to a median risk factor in US2AxiomaSH. Despite our earnest attempt to circumvent all possible sources of misalignment, the orthogonal component of implied alpha still manages to take exposure to latent systematic risk factors thus exposing the optimal portfolio to vagaries of FAP. How do we explain this phenomenon?

The answer to the above question lies in the following construction of implied alpha,

$$\gamma = \alpha - A^T \pi,$$

where $\gamma$ denotes implied alpha, $\alpha$ denotes expected returns (alpha), $A$ denotes the exposure matrix of constraints and $\pi$ denotes the associated shadow prices; the relationship expressed in the above equation naturally carries over to the orthogonal components of implied alpha, alpha and constraint exposures. Since we use identical alpha and risk factors, $\gamma_\perp$ is determined by the orthogonal component of constraints. Furthermore, since industry and sector bound constraints are derived from industry factors in US2AxiomaSH, the only constraints that have a non-trivial orthogonal component are the long-only and active asset bound constraints given by,

$$h_i \geq 0 \quad \text{(long-only constraint)}$$

$$h_i - b_i \geq l_i \quad \text{(active lower bound constraint)}$$

$$h_i - b_i \leq u_i \quad \text{(active upper bound constraint)} ;$$

$h_i$ and $b_i$ denote the portfolio and benchmark holding for asset $i$, respectively. During the process of portfolio construction, the optimizer determines the subset of long-only and active asset bound constraints that would be binding at the optimal portfolio. If the set of assets corresponding to these binding constraints have certain common aka ‘systemic’ characteristics then there is a possibility for the existence of latent systematic risk exposure in the resulting portfolios. We illustrate this subtle but extremely important technical point by two examples.

First, consider a Socially Responsible Investment (SRI) strategy that prohibits investment in companies that benefit from tobacco, alcohol or gambling activities; for sake of brevity, we refer to such companies as TAG companies. Naturally optimal portfolios managed according to the SRI mandate would have negative active exposure to TAG stocks. If the risk model that is used in construction of such SRI portfolios does not have a TAG industry factor then the optimizer is likely to overlook the systematic component of the negative active TAG exposure resulting in unaccounted systematic risk and a downward bias in risk prediction. Note that in this case the misalignment is introduced by binding constraints that prohibit
the ownership of TAG stocks regardless of the mutual alignment, or lack thereof, between the alpha and risk factors.

Second, in the context of our base strategy, note that long-only and active asset bound constraints are likely to be binding at assets that are either in the bottom or top decile of the alpha ranking. Such a collection of assets can have its unique characteristics and hence systematic risk exposures which are not completely captured by the risk model despite conformity between the alpha and risk factors. In order to test this hypothesis, we conducted the following experiment.

We derived the so-called “tail-alpha” from the alpha for the above strategy by sorting the assets by their alpha scores and setting the alpha exposure of all assets which are not present in the top or bottom decile to zero. To put it differently, we retained the original alpha values for only those assets that are most likely to yield binding active asset bound or long-only constraints. Figure 2a shows the decomposition of tail alpha in terms of the spanned and orthogonal components while Figure 2b shows the realized systematic risk of the orthogonal component of tail alpha computed using the augmented regression model. It is interesting to note that even though tail alpha is derived using risk factors in US2AxiomaSH, it still has non-trivial orthogonal component which in turn has significant latent systematic risk.

These experiments illustrate how long-only and active asset bound constraints introduce misalignment in a strategy that uses identical alpha and risk models. Next, we proceed to analyze the marginal impact of various practical considerations on these findings. One of the common features of almost every quantitative strategy is a conscious attempt to control transaction costs and other trading expenses. As show in Figure 1d, the monthly turnover of portfolios generated using the base strategy was unreasonably high. We conducted an alternative set of experiments wherein we augmented the original strategy by a turnover constraint that limits the monthly turnover to 18%. Figures 3a-3d report the key findings of this experiment. Note that the qualitative nature of these results - statistically significant downward bias in risk prediction and latent systematic risk in the orthogonal component of implied alpha - is similar to those obtained with the base strategy. The humped shape of the bias statistic graph shown in Figure 3b suggests that the bias in risk prediction initially increases with increasing active risk targets and then continues to decline. The section that follows provides a theoretical model to explain this rather intriguing phenomenon.

Some portfolio managers (PM) prefer to optimize the trade off between expected returns and transaction costs as a proxy for the turnover constraint. To capture the effect of misalignment on such strategies, we modified the objective function of the base strategy to include a transaction cost term. Figures 4a-4d report the computational results reconfirming the downward bias and other symptoms of FAP despite complete alignment between alpha and risk models. These two cases are particularly interesting since they show how misalignment can arise from trading considerations.

Another important concern while managing small-cap portfolios is excessive exposure to illiquid stocks. Markets for such stocks have limited depth, and excessive trading, measured as a proportion of average daily volume (ADV), can significantly move the stock prices. There are several ways of incorporating these liquidity concerns; we discuss three such approaches below in the context of FAP.

In the first set of experiments, we added a market impact function to the objective function of the base strategy to capture the effect of trading induced price fluctuations.
Figures 5a-5d report our key findings. It is interesting to note that the bias statistics in this case are close to 1.4 for some of the risk targets thus defying the notion that only alphas contribute to misalignment problems. A second approach to handle liquidity concerns is to add a constraint, referred to as the ADV constraint, that limits the exposure to assets with limited average daily trading volume; Figures 6a-6d reports the results for the strategy obtained by adding the ADV constraint to the base strategy. Finally, a third approach entails modifying the active asset bounds for illiquid assets. We implemented this approach by reducing the maximum possible deviation from benchmark weights by 50% for assets which were in the bottommost quartile of the liquidity factor in US2AxiomaSH; Figures 7a-7d report the computational results.

One may be tempted to believe that misalignment problems illustrated so far are limited to quantitative strategies that invest exclusively in small-cap assets or use a short-horizon risk model. We next describe a simple experiment that refutes this hypothesis. We repeated the above set of experiments with a variant of the base strategy obtained by making three modifications. First, we replaced the benchmark, and hence the universe of investable assets, by S&P 500. Second, we used the medium horizon risk model (US2AxiomaMH) instead of US2AxiomaSH. Finally, we used the medium-term variant of the momentum factor instead of the short-term variant to define the expected returns. Figures 8a-8d report the key results. Symptoms of FAP that we discovered in previous set of experiments are equally conspicuous in this case too, our choice of a large-cap asset universe or a different risk model notwithstanding.

A common theme among all of these experiments is the empirical demonstration of FAP despite using a common set of alpha and risk factors. The strategies used in these experiments were intentionally kept simple to facilitate effortless replication; we believe that similar results would be obtained regardless of the choice of optimizer or risk model. The section that follows provides a theoretical underpinning for the empirical results presented in this section.

3 Theoretical Insights

In this section, we give some theoretical results to shed light on the role of constraints in the context of factor alignment problems. For the sake of analytical accessibility, we limit our discussion to constrained mean-variance optimization (MVO) problems with a single constraint. All of these results can be easily generalized to encompass several additional constraints.

Consider the following constrained MVO problem with a single factor exposure constraint,

\[
\begin{align*}
\text{max} & \quad \alpha^T h - \frac{1}{2} h^T Q h \\
\text{s.t.} & \quad \beta^T h \geq \beta_0 .
\end{align*}
\]

Let \( h(\alpha, \beta) \) denote the optimal solution to \( \text{MVO}(\alpha, \beta) \). Our goal is to understand the role of the constraint \( \beta^T h \geq \beta_0 \) in influencing the composition of the optimal portfolio \( h(\alpha, \beta) \). In order to pursue this goal, we define two auxiliary unconstrained MVO problems, namely,

\[
\begin{align*}
\text{max} & \quad \alpha^T h - \frac{1}{2} h^T Q h \quad \text{MVO}(\alpha) \quad \text{and} \\
\text{max} & \quad \beta^T h - \frac{1}{2} h^T Q h \quad \text{MVO}(\beta) .
\end{align*}
\]
Let $h(\alpha)$ and $h(\beta)$ denote the optimal solutions to $MVO(\alpha)$ and $MVO(\beta)$, respectively. Note that if $\beta^T h(\alpha) \geq \beta_0$ then $h(\alpha)$ is also an optimal solution to $MVO(\alpha, \beta)$ thereby rendering the constraint $\beta^T h \geq \beta_0$ irrelevant. Thus for the purpose of our discussion we assume that $h(\alpha)$ violates the constraint $\beta^T h \geq \beta_0$, and let $\eta = \beta_0 - \beta^T h(\alpha)$ denote the associated constraint violation. Furthermore, without loss of generality we can assume that $\beta^T Q^{-1} \beta = 1$. The theorem that follows establishes an important link between $h(\alpha)$, $h(\beta)$ and $h(\alpha, \beta)$.

**Theorem 1.** $h(\alpha, \beta) = h(\alpha) + (\eta \lambda) h(\beta)$.

Theorem 1 shows that the optimal solution to $MVO(\alpha, \beta)$ is obtained by tilting the optimal solution $h(\alpha)$ to the unconstrained problem $MVO(\alpha)$ in the direction $h(\beta)$. Furthermore, the extent of tilting is jointly determined by the risk aversion parameter $\lambda$ in $MVO(\alpha, \beta)$ and the violation $\eta$ of the constraint $\beta^T h \geq \beta_0$ by $h(\alpha)$. The higher the risk aversion parameter $\lambda$, more significant is the influence of the constraint $\beta^T h \geq \beta_0$ in determining $h(\alpha, \beta)$. Similarly, tighter constraints give rise to higher violation $\eta$ and consequently have greater influence in determining $h(\alpha, \beta)$.

Theorem 1 also provides additional insights from an alignment perspective. Note that the relationship expressed in Theorem 1 naturally extends to the orthogonal components of $h(\alpha)$, $h(\beta)$ and $h(\alpha, \beta)$. It has been well documented in the literature (Lee and Stefek (2008); Saxena and Stubbs (2010b)) that optimal portfolios associated with unconstrained MVO problems load up on the orthogonal component of the expected returns. For instance, if $\alpha_\perp \neq 0 \ (\beta_\perp \neq 0)$ then $h(\alpha) \ (h(\beta))$ will have disproportionately higher exposure to $\alpha_\perp (\beta_\perp)$. Theorem 1 extends these findings to constrained MVO problems with an intriguing twist. It shows that $h(\alpha, \beta)$ loads up not only on the orthogonal component of $\alpha$, by virtue of the term $h(\alpha)$, but also on the orthogonal component of $\beta$ due to the presence of the term $(\eta \lambda) h(\beta)$. Furthermore, the extent of overloading depends directly on the magnitudes of $\lambda$ and $\eta$. Specifically, highly risk averse strategies that use a higher value of $\lambda$, or equivalently lower value of risk targets $\sigma$, are more likely to suffer from misalignment arising from constraints. Of course, if the value of $\lambda \ (\sigma)$ is very large (small) then the role of constraints diminishes and the portfolio holdings start to resemble minimum variance portfolios, or the benchmark holdings in the case of active strategies.

To summarize, the downward bias in risk prediction that arises exclusively due to the presence of constraints should have a humped shape attaining highest values at moderate risk target levels. This jibes well with the shapes of bias statistics charts shown in Figures 1b, 3b, 4b, 5b, 6b and 7b. By similar arguments, it follows that strategies with tighter constraints leading to higher values of the violation parameter $(\eta)$ would betray similar characteristics.

Until now we have examined results that corroborate the role of constraints in the construction of optimal portfolios. Next we present an interesting result that reverses the roles of alphas and constraints altogether. Consider the following MVO problem.

$$\max_{\alpha} \quad \beta^T h - \frac{1}{2\eta} h^T Q h$$

$$\text{s.t.} \quad \alpha^T h \geq \alpha_0,$$

$$MVO(\beta, \alpha),$$

where $\alpha_0 = \alpha^T h(\alpha, \beta)$. Let $h(\beta, \alpha)$ denote the optimal solution to $MVO(\beta, \alpha)$.

**Theorem 2.** ($\alpha - \beta$ Interchangeability Theorem) $h(\alpha, \beta) = h(\beta, \alpha)$. 
Theorem 2 shows that \( MVO(\alpha, \beta) \) and \( MVO(\beta, \alpha) \) have identical optimal solutions. In other words, there is nothing sacrosanct about alphas in a constrained MVO problem, and the same optimal portfolio can be obtained by switching the role of alphas and constraints. As an immediate corollary, it follows that the misalignment between constraints and risk factors can have as much influence, if not more, in determining the composition of optimal holdings as that between alpha and risk factors. Furthermore, the relative significance of misalignment due to alpha and constraints can be gauged by comparing the risk-aversion parameters in \( MVO(\alpha, \beta) \) and \( MVO(\beta, \alpha) \). Specifically, the higher the violation \( \eta \), the smaller is the risk aversion parameter in \( MVO(\beta, \alpha) \) and more prominent is the role of constraints. Notably, the ratio of the risk aversion parameters, namely \( \frac{\eta \lambda}{\gamma} \), is precisely the amount by which \( h(\alpha) \) is tilted towards \( h(\beta) \) to determine the optimal solution to \( MVO(\alpha, \beta) \) (see Theorem 1).

Next we briefly discuss a solution approach, namely the Alpha Alignment Factor (AAF) methodology, to address misalignment arising from constraints. We limit our discussion to key insights and refer the readers to Saxena and Stubbs (2010b) for further details. Since the focus of this paper is misalignment arising exclusively from constraints, we assume that \( \alpha_{\perp} = 0 \) in the discussion that follows. Recall that if \( \alpha_{\perp} = 0 \), then the only source of misalignment is the orthogonal component of \( \beta \). In fact, in this case it can be easily shown that the orthogonal component of implied alpha (\( \gamma \)) and \( \beta_{\perp} \) point in the same direction i.e. \( \frac{1}{\|\beta_{\perp}\|} \gamma = \frac{1}{\|\beta_{\perp}\|} \beta_{\perp}. \) The AAF approach recognizes the possibility of systematic risk in the orthogonal component of implied alpha, and penalizes the exposure of the portfolio to \( \gamma_{\perp} \).

In our special setting, the AAF optimization problem can be stated as,

\[
\begin{aligned}
\text{max} & \quad \alpha^TH - \frac{1}{2} \left( \beta^T Q \beta + \nu (h^T y)^2 \right) \\
\text{s.t.} & \quad \beta^T \nu \geq \beta_0 ,
\end{aligned}
\]

where \( y = \frac{1}{\|\beta_{\perp}\|} \beta_{\perp}\), and \( \nu \) is the systematic risk associated with \( y \). Note that \( MVO(AAF) \) can be obtained from \( MVO(\alpha, \beta) \) by replacing the covariance matrix \( Q \) by an augmented covariance matrix \( Q_y = QQ + \nu yy^T \) that has an additional variance term \( \nu yy^T \) to capture systematic risks in portfolios by virtue of exposure to \( \beta_{\perp} \). Net we discuss some important characteristics of the optimal solution, say \( h_y \), to \( MVO(AAF) \), and compare them with those of \( h(\alpha, \beta) \).

Under certain assumptions as laid out in Saxena and Stubbs (2010b), it can be shown that the predicted risk of \( h_y \), namely \( \sqrt{h_y^T Q h_y} \), is an unbiased estimate of the realized risk of \( h_y \). In other words, while solving \( MVO(AAF) \) the optimizer uses an unbiased risk estimate while choosing the optimal portfolio. The same cannot be said about \( MVO(\alpha) \). Since the systematic risk of \( h(\alpha, \beta) \) that arises by virtue of exposure to \( \beta_{\perp} \) is not captured by \( Q \), and hence goes unaccounted during the optimization phase, it follows that the optimizer’s ability to select portfolios that have optimal ex-post risk adjusted performance is severely curtailed while solving \( MVO(\alpha, \beta) \). This statement can be made precise by using the concept of utility function as described below (see Saxena and Stubbs (2010b) for further details).

Let \( U(h) = \alpha^T h - \frac{1}{2} \sigma^2(h) \) denote the utility function associated with an arbitrary portfolio \( h \); \( \sigma(h) \) denotes the ‘realized’ risk of \( h \). It can be shown that \( U(h_y) \geq U(h(\alpha, \beta)) \), and the inequality is strict provided \( \beta_X \neq 0 \) and \( \nu > 0 \). Thus using the AAF approach not only gives unbiased risk estimates but also improves the ex-post utility function. Phrased using
the concept of efficient frontiers, AAF approach pushes the ex-post frontier upwards thereby allowing the PM to access portfolios that lie above the traditional efficient frontier. The section that follows illustrates this “pushing frontier” phenomenon using the USER model (see Guerard et al.). To summarize, misalignment arising from constraints is as important and harmful as that arising from misaligned alpha factors. It not only creates statistically significant biases in risk prediction but also obfuscates the ability of the optimizer to solve the quintessential asset allocation problem. AAF approach attacks this problem at its very core; it recognizes the existence of latent systematic risk factors, creates disincentives for the optimizer to load up on such factors, and delivers portfolios that not only have readily available unbiased risk estimates but also superior ex-post risk-adjusted performance.

We conclude this section on an important practical note. Admittedly, the violation parameter \( \eta \) plays a very important role in the narrative presented above. We would like to remind the readers that the violation of constraints by optimal portfolios derived using the unconstrained MVO model is a very common phenomenon; such portfolios are often un-investable due to concentrated long/short positions in certain stocks, excessive turnover, violation of IPS mandates, unacceptable exposures to certain industries/sectors, or simply because they defy common wisdom. Thus constraints are an inextricable component of any quantitative strategy, and as illustrated by the results presented in this paper their contribution to FAP cannot be relegated to secondary considerations.

### 4 Opportunity Costs of FAP

Until now our discussion has been focused on illustrating the symptoms of FAP that arise by virtue of misalignment due to constraints. In this section we highlight the “opportunity” cost of not addressing the misalignment between constraints and risk factors. Our results show that the ill effects of misalignment due to constraints extend beyond the immediately visible effects such as biased risk prediction, and materially impact the efficiency of the resulting portfolios. In other words, the latent systematic risk factors associated with constraints disorient the ability of the optimizer to perform optimal budget and risk allocation as justified by the risk-return characteristics of individual securities. As a byproduct, we show how the Alpha Alignment Factor (AAF) approach remedies many of these problems and helps restore efficiency of the optimal portfolio; we refer the readers to Saxena and Stubbs (2010a,b); Ceria et al. (2012); Saxena and Stubbs (2011) for background on the AAF methodology.

We used the following strategy in our experiments,

\[
\begin{align*}
\text{maximize} & \quad \text{Expected Return} \\
\text{s.t.} & \\
& \text{Fully invested long-only portfolio} \\
& \text{Active GICS sector exposure constraint} \\
& \text{Active GICS industry exposure constraint} \\
& \text{Active asset bounds constraint} \\
& \text{Turnover constraint (two-way; 16\%)} \\
& \text{Active Risk constraint (}\sigma\text{)} \\
& \text{Benchmark = Russell 3000.}
\end{align*}
\]
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The expected returns were derived using the United States Expected Returns (USER) model (Guerard et al.). The above strategy is similar to the one used in Saxena and Stubbs (2011) with two key differences. First, we used a slightly tighter set of constraints for the sake of illustration. Second, instead of using Axioma’s fundamental medium horizon risk model (US2AxiomaMH), we used a custom risk model (CRM) derived from US2AxiomaMH by incorporating the USER expected returns as an additional risk factor. Our choice of CRM is motivated by the desire to isolate the effect of misalignment that arises exclusively due to the presence of constraints. Note that by construction, the alphas for the above strategy are completely spanned by the risk factors of CRM.

We ran monthly backtests using the above strategy in the 1999-2010 time period for $\sigma = 1.0\%, 1.1\%, \ldots, 5.0\%$. These backtests were run in two setups that were identical in all respects except that the first setup used the CRM alone while the second setup used the CRM in conjunction with the AAF. Figures 9a-9d report the key results. Using just the CRM produced a bias in the risk forecast (Figure 9a) that was statistically significant (Figure 9b) and could be attributed to latent systematic risk in the orthogonal component $\gamma_\perp$ of implied alpha (Figure 9c). Using the AAF approach alleviates all of these problems and yields unbiased risk prediction (Figures 9a and 9b); the AAF approach recognizes the possibility of systematic risk in $\gamma_\perp$ and accordingly penalizes the exposure of the portfolio to $\gamma_\perp$ (see Saxena and Stubbs (2010b) for further details).

Figure 9d reports the ex-post risk-return frontier. If the effect of misalignment due to constraints was limited to risk forecasting errors, then using the AAF approach should simply remove the bias in risk prediction and move the portfolio “on” the original frontier. However, as is evident from Figure 9d, using the AAF approach “pushed” the frontier upwards thereby producing improvement in risk adjusted performance besides the expected enhancements in accuracy of risk forecasting. Among other things, this implies that the original frontier was not efficient to start with. In other words, the misalignment arising from constraints was so severe that it compromised the optimality of the resulting portfolio, one of the primary goals of quantitative investing. Detailed investigation of this “pushing frontier” phenomenon goes beyond the scope of this paper and is addressed elsewhere (see Saxena and Stubbs (2010b, 2011)). We excerpt some of the arguments from Saxena and Stubbs (2011) to help readers understand the reasons of these improvements.

Risk models play a pivotal role in the construction of optimized portfolios. They assist the optimizer in striking an optimal balance between the temptation to overload on the alphas and thereby allocate the entire budget in reaping expected returns, and the necessity to satisfy the constraints specified in the investment strategy. Consequently, the influence of risk models is not simply limited to obtaining the ex ante risk forecasts. Instead, they materially affect the composition of optimal holdings, budget and risk allocation across various securities, turnover utilization, and primary characteristics of interest such as information ratio, Sharpe ratio, transfer coefficient, etc. Naturally, if there are systematic biases in the optimal portfolio that are not captured by the risk model, all of these mentioned characteristics get affected resulting in inefficient risk and budget allocation. By recognizing and correcting for the existence of unaccounted systematic risk factors arising from both misaligned alpha factors and constraints, the AAF approach makes holistic improvements to the process of portfolio construction resulting in not only better risk forecasts but also improved ex-post performance thereby restoring Markowitz’ MVO efficiency.
5 Conclusion

In this paper we set out to illustrate the role of constraints in introducing FAP despite complete conformity between alpha and risk factors. We presented several empirical and theoretical results to corroborate our point of view and in the process of doing so, produced research that helps to better understand the mechanics of quantitative portfolio construction. We close the paper with two concluding remarks.

First, even though the impact of constraints is difficult to comprehend, their role should not be under-emphasized or relegated to ancillary status during investigation of quantitative strategies. The key to understanding the influence of constraints lies in designing experiments that can isolate their effects and lend themselves to insightful analysis. The experiments discussed in Section 2 were motivated by these considerations. We encourage the readers to replicate these experiments and convince themselves that aligning alpha and risk factors is only a partial remedy to FAP. Second, we want to remind the readers that the alternative source of misalignment, namely misaligned alpha factors, is equally important and it is the combination of alpha and constraints that cause FAP. Fortunately, development of products such as the Risk Model Machine (RMM), have made it considerably easier to handle FAP due to misaligned alpha factors. Research efforts to leverage tools such as the RMM in handling misalignment arising due to constraints hold clues for further improvements.

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Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

(a) Realized vs predicted active risk

(b) Bias statistic

(c) Systematic risk in the orthogonal component (σ = 2%)

(d) Average turnover

Figure 1: Base Strategy Results

Figures
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

(a) Spanned vs Orthogonal Component

(b) Systematic risk in the orthogonal component

Figure 2: Alignment analysis of tail alpha

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component ($\sigma = 2\%$)

(d) Average turnover

Figure 3: Base Strategy with turnover constraint
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component ($\sigma = 2\%$)

(d) Average turnover

Figure 4: Base Strategy with transaction cost objective term
Figure 5: Base Strategy with market impact function objective term

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component ($\sigma = 2\%$)

(d) Average turnover
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component ($\sigma = 2\%$)

(d) Average turnover

Figure 6: Base Strategy with Average Daily Volume (ADV) constraint
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

Figure 7: Base Strategy with illiquidity motivated active asset bounds

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component ($\sigma = 2\%$)

(d) Average turnover
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

Figure 8: Variant of the base strategy that uses large-cap asset universe and medium-horizon fundamental risk model

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component (σ = 2%)

(d) Average turnover
Aligning Alpha and Risk Factors, a Panacea to Factor Alignment Problems?

(a) Realized vs predicted active risk

(b) Bias statistic (active risk)

(c) Systematic risk in the orthogonal component ($\sigma = 3\%$)

(d) Realized risk-return frontier

Figure 9: Case study using the USER model
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